Theorem (Intermediate Value Theorem)**.** *If f is continuous on $[a, b]$ and K is a number between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that,*

$$f(c) = K.$$

Proof. Let us assume without loss of generality that $f(a) > K$ and $f(b) < K$, the proof of the other case is identical. Now let us define,

$$[L_0, R_0] = [a, b].$$

Now for $n \geq 1$ we will define $[L_n, R_n]$ recursively by,

$$[L_n, R_n] = \begin{cases} [\frac{L_{n-1}+R_{n-1}}{2}, R_{n-1}] & \text{if } f\left(\frac{L_{n-1}+R_{n-1}}{2}\right) > K \\ [L_{n-1}, \frac{L_{n-1}+R_{n-1}}{2}] & \text{if } f\left(\frac{L_{n-1}+R_{n-1}}{2}\right) < K \end{cases}$$

In other words, we bisect the interval and pick it in a way such that $f(L_n) > K$ and $f(R_n) < K$ for all n. Note that L_n is an increasing sequence bounded by b, and R_n is decreasing sequence bounded by a, so by monotone convergence theorem, we have there is some L, R such that

$$\lim_{n\to\infty} L_n = L \quad \lim_{n\to\infty} R_n = R.$$

Since the legth of $[L_n, R_n]$ is half the length of the previous interval we have,

$$|R_n - L_n| = \frac{1}{2^n}|L_0 - R_0| = \frac{1}{2^n}(b - a) \to 0 \quad \text{as } n \to \infty.$$

So we must have ,

$$L = R \equiv c.$$

It remains to show that $f(c) = K$. Note that this is the same as saying for all $\varepsilon > 0$,

$$f(c) - \varepsilon < K < f(c) + \varepsilon.$$

Since f is continuous at c, we have there is a $\delta > 0$ such that whenever $c - \delta < x < c + \delta$, we have $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. Since L_n and R_n converge to c, we have that there is a N such that

$$c - \delta < L_N, R_N < c + \delta.$$

So we have

$$f(c) - \varepsilon < f(L_N) < K < f(R_N) < f(c) + \varepsilon.$$

Since this is true for all ε, we must have $f(c) = K$. □